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MATHEMATICS

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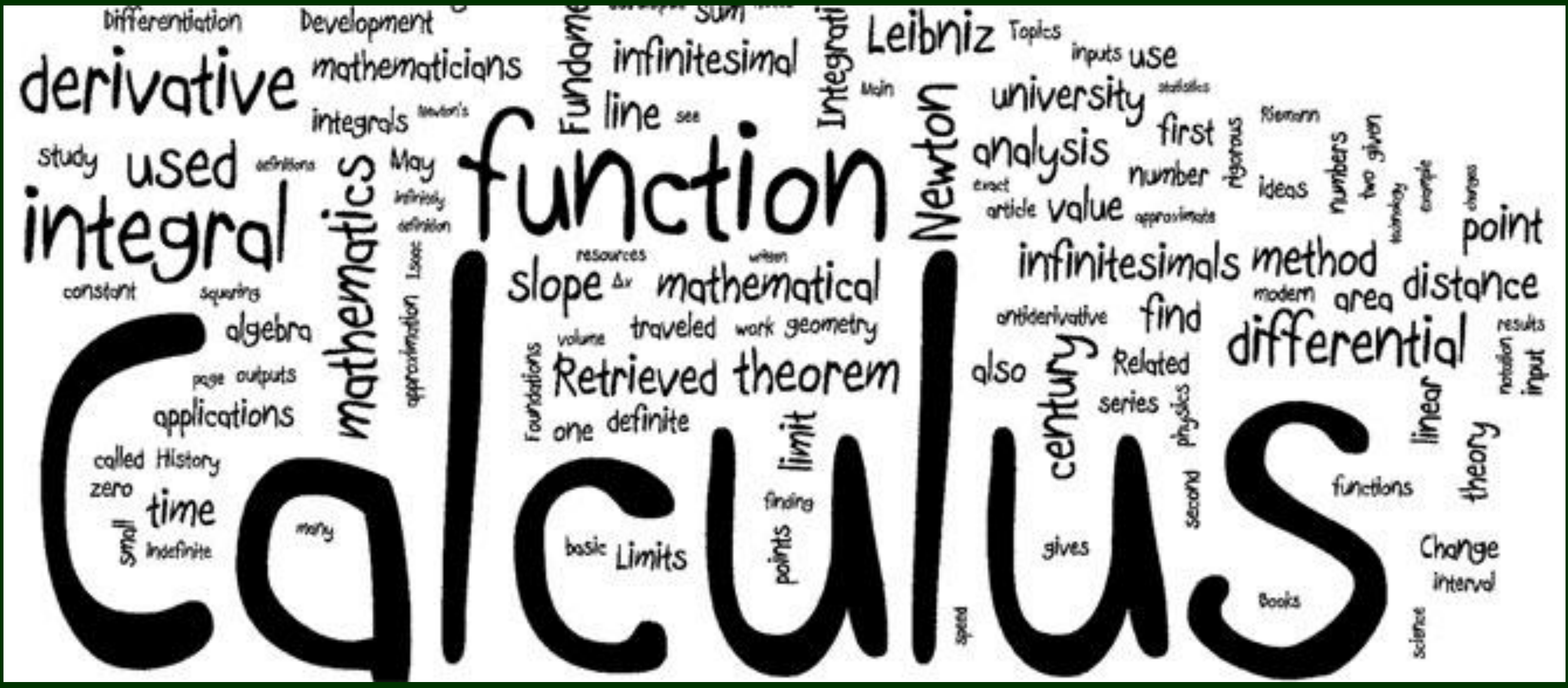

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Third Edition

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GATE 2020

ORGANIZING INSTITUTE :

IIT DELHI

AEROSPACE ENGINEERING

Mathematics Syllabus

Core Topics:

Linear Algebra: Vector algebra, Matrix algebra, systems of linear equations, rank of a matrix, eigen values and eigen vectors.

Calculus: Functions of single variable, limits, continuity and differentiability, mean value theorem, chain rule, partial derivatives, maxima and minima, gradient, divergence and curl, directional derivatives. Integration, Line, surface and volume integrals. Theorems of Stokes, Gauss and Green.

Differential Equations: First order linear and nonlinear differential equations, higher order linear ODEs with constant coefficients. Partial differential equations and separation of variables methods.

Special Topics: Fourier Series, Laplace Transforms, Numerical methods for linear and nonlinear algebraic equations, Numerical integration and differentiation.

Mathematics Year Wise Analysis

Year	No of Questions	Topics (1 marks + 2 marks)	Total Marks
2019	1M : 5 2M : 4	• Linear Algebra (0+2) • Differential Calculus (3+2) • Vector Calculus (2+0)	13
2018	1M : 4 2M : 3	• Linear Algebra (1+0) • Differential Calculus (1+0) • Vector Calculus (2+2) • ODE (0+1)	10
2017	1M : 5 2M : 5	• Linear Algebra (0+2) • Vector Calculus (3+0) • ODE (1+1) • PDE (0+1) • Numerical Technique (1+0) • Laplace (0+1)	15
2016	1M : 5 2M : 4	• Linear Algebra (2+1) • Differential Calculus (1+1) • Vector Calculus (1+0) • ODE (0+1) • PDE (1+0) • Numerical Technique (0+1)	13
2015	1M : 3 2M : 5	• Linear Algebra (1+1) • Differential Calculus (1+1) • ODE (0+2) • PDE (1+0) • Numerical Technique (0+1)	13
2014	1M : 5 2M : 5	• Linear Algebra (1+1) • Differential Calculus (1+0) • Vector Calculus (0+1) • ODE (1+1) • Numerical Technique (1+0) • Laplace (0+1) • Fourier Series(0+1) • Series(1+0)	15
2013	1M : 4 2M : 4	• Linear Algebra (1+1) • Differential Calculus (2+0) • Vector Calculus (1+1)	12

		• ODE (0+1) • Laplace (0+1)	
2012	1M:5 2M:4	• Linear Algebra (2+1) • Differential Calculus (0+2) • ODE (2+0) • Numerical Technique (0+1) • Laplace (1+0)	13
2011	1M:5 2M:3	• Linear Algebra (1+1) • Differential Calculus (2+0) • Vector Calculus (2+0) • ODE (0+1) • Numerical Technique (0+1)	11
2010	1M:5 2M:5	• Linear Algebra (1+0) • Differential Calculus (1+2) • Vector Calculus (1+1) • ODE (1+0) • PDE (1+0) • Numerical Technique (0+1) • Laplace (0+1)	15
2009	1M:2 2M:6	• Linear Algebra (1+2) • Differential Calculus (0+1) • ODE (1+0) • PDE (0+0) • Numerical Technique (0+2) • Laplace (0+1)	14
2008 (85 questions)	1M:3 2M:9	• Linear Algebra (2+1) • Differential Calculus (0+1) • Vector Calculus (0+3) • ODE (1+1) • Numerical Technique (0+2) • Laplace (0+1)	21 (Total 150 marks)
2007 (85 questions)	1M:3 2M:12	• Linear Algebra (1+4) • Differential Calculus (1+1) • Vector Calculus (0+1) • ODE (0+2) • Numerical Technique (1+2) • Laplace (0+2)	27 (Total 150 marks)

TABLE OF CONTENT

- GATE and Practice questions
- Previous year GATE questions solved

Chapter 1

LINEAR ALGEBRA

The branch of mathematics that deals with the theory of systems of linear equations, matrices, vector spaces, and linear transformations.

1.1 MATRIX

A set *of* mn numbers (real or imaginary) arranged in the form of a rectangular array of *m* rows and *n* columns is called an *m x n* matrix (to be read as *'m* by *n'* matrix)

An *m x n* matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1j} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2j} \cdots a_{2n} \\ \vdots & \vdots & \vdots \quad \vdots \quad \vdots \\ a_{i1} & a_{i2} & a_{i3} \cdots a_{ij} \cdots a_{in} \\ \vdots & \vdots & \vdots \quad \vdots \quad \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots a_{mj} \cdots a_{mn} \end{bmatrix}$$

In compact form the above matrix is represented by

A = $[a_{ij}]_{m\times n}$ $\quad or \quad A = [a_{ij}]$

The numbers a_{11}, a_{12}, ... etc. are known as the elements of the matrix A. The element a belongs to ith row and jth column and is called the (*i, j*)th element of the matrix

$A = [a_{ij}]$ Thus, in the element a_{ij} , the first subscript *i* always denotes the number of rows and the second subscript *j*, denotes the number of columns in which the element occurs.

e.g., $\quad A = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$

is a matrix having 2 rows and 2 columns and so it is a matrix of order 2 x 2 such that a_{11} = sin x, a_{12} = cos x, b_{21} = cos x, b_{22}= - sin x

1.2 TYPE OF MATRICES

- **Row Matrix**

A matrix having only one row is called a row matrix or a row vector.

For example, A = [1 2 3 4] is a row matrix of order 1 x 4.

- **Column Matrix**

A matrix having only one column is called a column matrix or column vector.

For example, A = $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is column matrix of order 3 × 1.

- **Square Matrix**

A matrix in which the number of rows is equal to the number of columns, say *n* is called a square matrix of order *n.*

A square matrix of order *n* is also called a *n*-rowed square matrix. The elements a_{ij} of a square matrix $A = [a_{ij}]_{n\times n}$ for which *i* = *j*, the elements $a_{11}, a_{22}, \dots, a_{mn}$ are called the diagonal elements and the line along which they lie is called the principal diagonal or leading diagonal of the matrix.

For example, the matrix $\begin{bmatrix} 2 & 2 & 3 \\ 3 & -2 & 3 \\ 1 & 6 & -3 \end{bmatrix}$ is a square matrix of order 3 in which the diagonal elements are 2, -2 and -3.

- **Diagonal Matrix**

A square matrix $A = [a_{ij}]_{n\times n}$ is called a diagonal, if all the elements except those in the leading diagonal, are zero

i.e., a_{ij} = 0, ∀ i j

A diagonal matrix of order *n x n* having d_1 , d_2, ..., d_n as diagonal elements is denoted by diagonal [d_1 , d_2, ..., d_n]

For example, the matrix A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix to be denoted by A = diagonal (1,4,3].

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1.3 EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{r\times s}$ are equal, if

(i) m = r, the number of rows in *A* equals the number of rows in *B.*

(ii) *n= s i.*e., the number of columns in *A* equals the number of columns in *B.*

(iii*)* $a_{ij} = b_{ij}$ *f*or i = 1, 2, ..., m and j = 1, 2, ..., n. If two matrices A an*d B* are equal, we write *A = B,* otherwise we write A *B.*

Example 1.1: Find the values of x, y, z and *a* which satisfy the matrix equation.

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution: Since, the corresponding elements of two equal matrices are equal. Therefore,

x + 3 = 0, 2y + x = -7, z -1 = 3

and 4a – 6 = 2a

solving these equations, we get

a = 3, x = -3, y = -2, z = 4

1.26 SOLUTION OF LINEAR SYSTEM OF EQUATION

Cramer's Rule or Method of Determinant

Consider the equations

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_3 z = d_2$

$a_3 x + b_2 y + c_3 z = d_3$

If the determinant of coefficient be

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then, $x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $(\Delta \neq 0)$ ....(ii)

Similarly, $y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ ....(iii)

$z = \begin{vmatrix} d_1 & b_1 & d_1 \\ d_2 & b_2 & d_2 \\ d_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ ....(iv)

From Eqs. (ii), (iii) and (iv) giving the values of x, y and z constitute the cramer's rule

Example 1.10: Solve the equation 3x + y + 2z = 3, 2x – 3y – z = -3 and x + 2y + z = 4 by cramer's rule

Solution: here, $\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8$ (expanding by C_1)

Then, $x = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 1$ (expanding by C_1)

Similarly, $y = \frac{1}{\Delta}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 2$

And $z = \frac{1}{\Delta}\begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$

Hence, x = 1, y = 2 and z = -1.

1.27 CONSISTENCY OF LINEAR SYSTEM OF EQUATION

Consider the system of *m* linear equations.

$$\left.\begin{array}{c} a_{11}x_1 + x_{12}x_2 + \ldots + a_{1n}x_n = K_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = K_2 \\ : \quad : \quad : \quad : \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = K_n \end{array}\right\} \quad \ldots(i)$$

containing the n unknowns x_1, x_2,..., x_n. To determine whether the Eq. (i) are consistant (i.e., possess a solution) or not, we consider the rank of the matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

And
$$K = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & : & K_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & : & K_2 \\ \ldots & \ldots & \ldots & & \ldots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & : & K_N \end{bmatrix}$$

A is the coefficient marix and K is called the augmented of the Eq. (i).

Rouche's Theorem

The system of Eq. (i) is consistent if and only if the coefficient matrix A and the augmented matrix *K* are of the same rank, otherwise the system is inconsistent the solution will be unique only when $r = n$. Hence, the Eq. (1) are consistent.

Consistency of a System of Equations in n Unknowns (Matrix method)

Find the rank of the coefficient matrix A and the augmented matrix *K*, by reducing A to the triangular form by elementary row operations. Let the rank of A be r and that of *K* be r'

(i) If $r \neq r'$, the equations are inconsistent i.e., there is no solution.

(ii) If $r = r' = n$, the equations are consistent and there is a unique solution.

(iii) If $r = r' < n$, the equations are consistent and there are infinite number of solutions.

Giving arbitrary values to n - r of the unknowns we may express the other r unknowns in terms of these.

Example 1.12: Test for consistency any solve

5x + 3y +7z = 4,

3x + 26 y + 2z = 9,

7x + 2 y + 10z = 5.

Solution: We have, $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix} \qquad (3R_1, 5R_2)$$

$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix} \qquad (R_2, R_1)$$

$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix} \qquad \left(\frac{7}{3}R_1, 5R_3, \frac{1}{11}R_2\right)$$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \qquad \left(R_3 - R_1 + R_2, \frac{7}{3}R_1\right)$$

The rank of coefficient matrix and augmented matrix for the last set of equations are both 2. Hence, the equations are consistent. Also, the given system is equivalent to

5x + 3y + 7z = 4, 11y- z = 3

$y = \frac{3}{11} + \frac{z}{11}, x = \frac{7}{11} - \frac{16}{11}z$

where, z is a parameter.

Hence, $x = \frac{7}{11}, y = \frac{3}{11}$ and $z = 0$ is a particular solution

GATE QUESTIONS

Q1. The following system of equations

2x – y – z = 0

-x + 2y - z = 0

- x - y + 2z = 0

a) has no solution

b) has unique solution

c) has three solutions

d) has an infinite number of solutions. **[GATE 2019]**

Ans: d) has an infinite number of solutions.

Conditions: i) If D = 0, then the system of equations are consistent with infinite non- trivial solution

ii) If D ≠ 0, then the system of equations are consistent with trivial solution (x=0, y=0, z=0)

$$D = \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$

= 2(3) + 1(-3) – 1(1+2)

= 6 – 3 – 3 = 0

Therefore, infinite solutions

Q2. One of the eigen values of the following matrix is 1.

$$\begin{pmatrix} x & 2 \\ -1 & 3 \end{pmatrix}$$

The other eigen value is ______ **[GATE 2019]**

Ans: 2

$3 + x = 1 + \lambda_2$ [Sum of diagonal elements = sum of Eigen values]

$3x + 2 = \lambda_2$ [Determinant = product of Eigen values]

On solving,

$3(\lambda_2 - 2) + 2 = \lambda_2$

$3\lambda_2 - 6 + 2 = \lambda_2$

$2\lambda_2 = 4$

$\lambda_2 = 2$

Q3. The determinant of the $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & -3 \end{bmatrix}$ is _____ (accurate to one decimal place). **[GATE 2018]**

Ans: zero

PRACTICE QUESTIONS

Q1. Nullity *of* the matrix A = $\begin{bmatrix} -1 & 4 & 2 \\ 1 & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 6 & 4 \end{bmatrix}$ is

(a) 1

(b) 2

(c) 3

(d) 4

Ans: (a) 1

$A = \begin{bmatrix} -1 & 4 & 2 \\ 1 & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 6 & 4 \end{bmatrix}$ $(R_2 + R_1, R_3 + R_4)$

$= \begin{bmatrix} -1 & 4 & 2 \\ 0 & 7 & 4 \\ 0 & 7 & 4 \\ 2 & 6 & 4 \end{bmatrix}$ $(R_4 + 2R_1)$

$= \begin{bmatrix} -1 & 4 & 2 \\ 0 & 7 & 4 \\ 0 & 7 & 4 \\ 0 & 14 & 8 \end{bmatrix} = 2\begin{bmatrix} -1 & 4 & 2 \\ 0 & 7 & 4 \\ 0 & 7 & 4 \\ 0 & 7 & 4 \end{bmatrix} = 2\begin{bmatrix} -1 & 4 & 2 \\ 0 & 7 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, rank of matrix A = 2 = Number of non-zero rows

∴ Nullity of matrix A = Number of column of A - Rank of A

= 3 - 2 = 1

Q2. If A and *B* are real symmetric matrices of size *n x n,* then

(a) $AA^T = I$

(b) $A = A^{-1}$

(c) A*B* =*B*A

(d) $(AB)^T = B^T A^T$

Ans: (d) The transpose of the product of two matrices is the product in reverse order of their transposes (Reversal law for the transpose of a product).

Q3. Consider the following statements 2:

S_1 : Sum of the two singular matrices may be non-singular.

S_2 :Sum of the two non-singular n x n matrices may be singular.

Which of the following statements is correct?

Q13. How many solutions does the following system linear equations have?

-x + 5y = -1

x - y = 2

x + 3y = 3

(a) Infinitely many

(b) Two distinct solution

(c) Unique

(d) None who

Ans: (c) Unique

-x + 5y = -1 ...(i)

x – y = 2 ...(ii)

x + 3y = 3 ...(iii)

Solving Eqs. (i) and (ii), we get

x = $\frac{9}{4}$ and y = $\frac{1}{4}$

which satisfies Eq. (iii).

Hence, given system of linear equations have unique solution.

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Chapter 3

VECTOR CALCULUS

In general form vector can be written as

$$\bar{a} = |a| \cdot \hat{a} = a \cdot \hat{a}$$

In vector algebra

$$\bar{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} \qquad where, a_1, a_{2,}\ a_3\ are\ constant$$

While in vector calculus $a_1, a_2\ and\ a_3$ are variables

Eg : $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the radius vector

3.1 SCALAR PRODUCT (DOT PRODUCT)

$$\bar{a} \cdot \bar{b} = ab \cos\theta$$

Note: dot products always give scalar quantities

For orthogonal vectors, θ = 90°

$$\bar{a} \cdot \bar{b} = 0$$

$$i \cdot \hat{j} = i \cdot \hat{k} = j \cdot \hat{k} = 0$$

$$i \cdot \hat{\imath} = j \cdot \hat{j} = k \cdot \hat{k} = 1$$

3.2 CROSS PRODUCT (VECTOR PRODUCT)

$$\bar{a} \times \bar{b} = ab \sin\theta \cdot \hat{n}$$

Note: Cross products always gives always vector

$$\hat{\imath} \times \hat{\imath} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

$$i \times \hat{j} = \hat{j} \times \hat{k} = \hat{k} \times \hat{j} = 1$$

$$\hat{j} \times \hat{\imath} = -1$$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

3.3 DEL OPERATOR (∇)

∇ = $\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$

3.4 GRADIENT ∅ = (∇∅)

Where ∅ → scalar quantity

∇· ∅ = $\left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{j}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right) = \sum i\frac{\partial\emptyset}{\partial x}$

- gives vector as output

Eg. if ∅ = $x^2y + yz + xz$

Then ∇∅ = (2xy + z) $\hat{\imath}$ + $(x^2 + z)\hat{j} + (x + y)\hat{k}$

3.5 DIVERGENCE $\bar{F}$ $(\nabla \cdot \bar{F})$

Div$\bar{F}$ =$\nabla \cdot \bar{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(F_1\hat{\imath} + F_2\hat{j} + F_3\hat{k}\right)$

$\nabla \cdot \bar{F} = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) = \sum i \frac{\partial \bar{F}}{\partial x}$

Eg. If $\bar{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Then div $\bar{F} = 1 + 1 + 1$

$\nabla \cdot \bar{F} = 3$

NOTE: if divergence of 'F' is zero then the vector $\bar{F}$ is known as solenoidal vector

$$\nabla \cdot \bar{F} = 0$$

Example 3.1: If ∅ = $x^3 + y^3 + z^3 - 3xyz$, find $\nabla\emptyset, \nabla \cdot \nabla\emptyset, \nabla \times \nabla\emptyset$ at (1, 2, 3)

Solution: $\nabla\emptyset = x^3 + y^3 + z^3 - 3xyz$

$\nabla \cdot \emptyset = (3x^2 - 3yz)\hat{\imath} + (3y^2 - 3xz)\hat{\jmath} + (3z^2 - 3xy)\hat{k}$

$(\nabla\emptyset)_{1,2,3} = (3 - 9)\hat{\imath} + (12 - 9)\hat{\jmath} + (27 - 6)\hat{k}$

$(\nabla\emptyset)_{1,2,3} = -15\hat{\imath} + 3\hat{\jmath} + 21\hat{k}$

$\nabla \cdot \nabla\emptyset = 6x\hat{\imath} + 6y\hat{\jmath} + 6z\hat{k}$

$(\nabla\emptyset)_{1,2,3} = 6\hat{\imath} + 12\hat{\jmath} + 18\hat{k}$

$$\nabla \times \nabla\emptyset = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

= $\hat{\imath}(-3x + 3x) - \hat{\jmath}(-3y - 3y) + \hat{k}(-3z + 3z)$

= 0

3.6 CURL $\cdot \bar{F}$

Curl $\cdot \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \sum i \times \frac{\partial f}{\partial x}$

NOTE: if Curl $\bar{F} = 0$ then $\bar{F}$ is an irrotational vector. It can be represented as $\bar{F} = \nabla\ \emptyset$ (where ∅ is known as scalar potential)

Example 3.2: Prove that ∇∅ is an irrotational value

Solution:

$$\nabla \times \nabla\emptyset = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} & \frac{\partial\theta}{\partial z} \end{vmatrix}$$

$= \hat{\imath}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \hat{\jmath}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)$

= 0

3.12 INTEGRAL THEOREM

(1) Green's Theorem

If P, Q are the function of x, y and having continuous partial derivatives in a region r enclosed by a closed curve C then

$$\oint_c (P \cdot dx + Q \cdot dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

- Green's theorem in plane is a scalar of stokes theorem.

(2) Stroke's Theorem

If S is an open surface enclosed by a closed curve C and $\bar{F}$ is a vector point function having continuous partial derivatives then

$$\oint_c \bar{F} \cdot d\bar{r} = \iint_s curl\ \bar{F} \cdot d\bar{s} = \iint_s curl\ \bar{F} \cdot \hat{n} \cdot ds$$

- This is the relationship between line integral and surface integral

(3) Gauss Divergence Theorem

If S is an close surface of the region and closing a volume V and $\bar{F}$ has continuous partial derivatives then

$$\iint \bar{F} \cdot \hat{n} \cdot ds = \iiint_v div\bar{F} \cdot dx\, dy\, dz$$

- This is a relationship between a surface integral and volume integral
- Gauss divergence theorem is referred to as Green's theorem in space

(4) Scalar form of Gauss Divergence Theorm

$$\iint_s Pdydz + Qdxdz + Rdxdy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

GATE QUESTIONS

Q1. Vector $\vec{b}$ is obtained by rotating $\vec{a} = \hat{\imath} + \hat{\jmath}$ by 90° about $\hat{k}$, where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vectors along the x, y and z axes, respectively. $\vec{b}$ is given by

a) $\hat{\imath} - \hat{\jmath}$

b) $-\hat{\imath} + \hat{\jmath}$

c) $\hat{\imath} + \hat{\jmath}$

d) $-\hat{\imath} - \hat{\jmath}$ **[GATE 2019]**

Ans: b) $-\hat{\imath} + \hat{\jmath}$

Given: $\vec{a} = \hat{\imath} + \hat{\jmath}$

$$\vec{b} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= -\hat{\imath} + \hat{\jmath}$$

Q2. A scalar function is given by $f(x,y) = x^2 + y^2$. Take $\hat{\imath}$ and $\hat{\jmath}$ as unit vectors along the x and y axes, respectively. At (x,y) = (3, 4), the direction along which f increases the fastest is

a) $\frac{1}{5}(4\hat{\imath} - 3\hat{\jmath})$

b) $\frac{1}{5}(3\hat{\imath} - 4\hat{\jmath})$

c) $\frac{1}{5}(3\hat{\imath} + 4\hat{\jmath})$

d) $\frac{1}{5}(4\hat{\imath} + 3\hat{\jmath})$ **[GATE 2019]**

Ans: c) $\frac{1}{5}(3\hat{\imath} + 4\hat{\jmath})$

Given: $f(x, y) = x^2 + y^2$

$$\nabla f = 2x\hat{\imath} + 2y\hat{\jmath}$$

$$\nabla f\,|_{(3,4)} = 6\hat{\imath} + 8\hat{\jmath}$$

$$(\nabla\hat{f}) = \frac{6\hat{\imath} + 8\hat{\jmath}}{\sqrt{36 + 64}}$$

$$= \frac{6\hat{\imath} + 8\hat{\jmath}}{10}$$

$$= \frac{1}{5}(3\hat{\imath} + 4\hat{\jmath})$$

Chapter 8

FOURIER SERIES

8.1 CONDITION FOR A FOURIER EXPANSION

Any function f(x) can be developed as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_0 , a_n and b_n are constants provided.

(i) f(x) is periodic, single valued and finite.

(ii) f(x) has a finite number of discontinuities in any one period.

(iii) f(x) has at the most a finite number of maxima and minima.

8.2 IMPORTANT FORMULAE

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\ dx}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\ dx}$$

Example 8.1: Obtain the Fourier series for f(x) = e^{-x} in the interval 0 < x < 2π

Solution:

$e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ... (i)

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx}$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\, dx = \frac{1}{\pi}\ |-e^{-x}|_0^{2\pi}$$

$$= \frac{1 - e^{-2\pi}}{\pi}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\ dx}$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} \cos nx \; dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x} \; (-\cos nx + n \sin nx]_{\mathbf{0}}^{2\pi}$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right) \cdot \frac{1}{(n^2+1)}$$

$$\therefore \quad a_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2}, a_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{5} \text{ etc}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \; dx}$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x} \sin nx \; dx$$

$$= \frac{1}{\pi(n^2+1)}[e^{-x} \; (-\sin nx + n \cos nx]_{\mathbf{0}}^{2\pi}$$

$$= \left(\frac{1-e^{-2\pi}}{\pi}\right) \frac{n}{(n^2+1)}$$

$$\therefore \quad b_1 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{2}, b_2 = \left(\frac{1-e^{-2\pi}}{\pi}\right)\frac{1}{5} \text{ etc}$$

Substituting the values of a_0 , a_n and b_n in Eq. (i), we get

$$\boldsymbol{e^{-x} = \frac{1-e^{-2\pi}}{\pi}\left\{\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \cdots\right) + \left(\frac{1}{2}\sin x + \frac{1}{5}\sin 2x + \frac{1}{10}\sin 3x + \cdots\right)\right\}}$$

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